

$SL(2; R)$ Duality of the Noncommutative DBI Lagrangian

Davoud Kamani¹

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We study the action of the $SL(2; R)$ group on the noncommutative DBI Lagrangian. The symmetry conditions of this theory under the above group will be obtained. These conditions determine the extra $U(1)$ gauge field. By introducing some consistent relations we observe that the noncommutative (or ordinary) DBI Lagrangian and its $SL(2; R)$ dual theory are dual of each other. Therefore, we find some $SL(2; R)$ invariant equations. In this case the noncommutativity parameter, its T -dual and its $SL(2; R)$ dual versions are expressed in terms of each other. Furthermore, we show that on the effective variables, T -duality and $SL(2; R)$ duality do not commute. We also study the effects of the $SL(2; R)$ group on the noncommutative Chern–Simons action.

KEY WORDS: string theory; noncommutativity; $SL(2; R)$ duality.

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1. INTRODUCTION

$SL(2; R)$ duality generalizes strong-weak coupling duality. There is an $SL(2; R)$ symmetry manifest in the low energy action, which is broken down to $SL(2; Z)$ in string theory. Also there is considerable evidence in favor of this duality being an exact symmetry of the full string theory (Rey, 1991; Font *et al.*, 1990; Duff and Khuri, 1994; Sen, 1993,b,c; Schwarz and Sen, 1993; Duff and Rahmfeld, 1995; Hull and Townsend, 1995; Sen, 1994; Tseytlin, 1996). In fact, the $SL(2; R)$ group and its subgroup $SL(2; Z)$ act as symmetry groups of many theories (Green and Gutperle, 1996; Witten, 1996; Dc Roo, 1985; Schwarz, 1995; Gibbons and Rasheed, 1996; Cederwall and Townsend, 1997; Sen, 1994; Rey and von Unge, 2001; Gopakumar *et al.*, 2000; Lu *et al.*, 2000, 2001; Berman, 1997; Hofman and Verlinde, 1998; Chan *et al.*, 2001; Cai and Ohta, 2000). Among these theories, the noncommutative theories and the Dirac–Born–Infeld (DBI) theory are more important, for example see the Refs. (Rey and von Unge, 2001; Gopakumar *et al.*, 2000; Lu *et al.*, 2000, 2001; Berman, 1997; Hofman and Verlinde, 1998; Chan

¹Institute for Studies in Theoretical Physics and Mathematics (IPM), P.O. Box: 19395-5531, Tehran, Iran; e-mail: kamani@theory.ipm.ac.ir.

et al., 2001; Cai and Ohta, 2000; Born and Infeld, 1934; Leigh, 1989; Tseytlin, 1986, 1997; Fradkin and Tseytlin, 1985; Callan *et al.*, 1987; Abouelsaood *et al.*, 1987).

Consider the $SL(2; R)$ symmetry of the type IIB superstring theory (Rey, 1991; Font *et al.*, 1990; Duff and Khuri, 1994; Sen, 1993,b,c; Schwarz and Sen, 1993; Duff and Rahmfeld, 1995; Hull and Townsend, 1995; Sen, 1994; Tseytlin, 1996). In the type IIB theory the R-R zero-form χ and the dilaton ϕ of the NS-NS sector define a complex variable $\lambda = \chi + ie^{-\phi}$. Under the $SL(2; R)$ duality this variable and also the NS-NS and R-R two-forms $B_{\mu\nu}$ and $C_{\mu\nu}$ transform as in the following

$$\lambda \rightarrow \tilde{\lambda} = \frac{a\lambda + b}{c\lambda + d}, \quad \begin{pmatrix} B_{\mu\nu} \\ C_{\mu\nu} \end{pmatrix} \rightarrow \begin{pmatrix} \tilde{B}_{\mu\nu} \\ \tilde{C}_{\mu\nu} \end{pmatrix} = (\Lambda^T)^{-1} \begin{pmatrix} B_{\mu\nu} \\ C_{\mu\nu} \end{pmatrix},$$

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2; R). \quad (1)$$

In addition, the Einstein metric $g_{\mu\nu}^{(E)} = e^{-\phi/2}g_{\mu\nu}$ remains invariant. Therefore, the string coupling constant $g_s = e^\phi$ and the string metric $g_{\mu\nu}$ transform as follows

$$g_s \rightarrow \tilde{g}_s = \eta^2 g_s, \quad g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \eta g_{\mu\nu}, \quad \eta \equiv |c\lambda + d|. \quad (2)$$

For slowly varying fields, the effective Lagrangian of the open string theory is the DBI Lagrangian. For a review of this theory see Ref. (Born and Infeld, 1934; Leigh, 1989; Tseytlin, 1986, 1997; Fradkin and Tseytlin, 1985; Callan *et al.*, 1987; Abouelsaood *et al.*, 1987) and references therein. The equivalence of the noncommutative and ordinary DBI theories has been proven (Seiberg and Witten, 1999). We shall concentrate on both of these theories.

In Section 2, we shall present an $SL(2; R)$ invariant argument for the ordinary and noncommutative DBI Lagrangians. Therefore, for special $C_{\mu\nu}$ a Dp -brane with ordinary worldvolume, but modified tension will be obtained. In addition, we obtain the auxiliary $U(1)$ gauge field strength $\tilde{F}_{\mu\nu}$ (Townsend, 1997) in terms of the other variables. This field with the $U(1)$ field strength $F_{\mu\nu}$ form an $SL(2; R)$ doublet.

In Section 3, by introducing a consistent relation between $B_{\mu\nu}$ and $\tilde{B}_{\mu\nu}$, a useful rule will be obtained. That is, the DBI theory and its $SL(2; R)$ dual theory are duals of each other. In other words, twice dualizing of the DBI theory leaves it invariant. This reflection also holds for the noncommutative DBI theory.

In Section 4, we shall obtain some relations between the effective open string variables and their duals. Thus, $SL(2; R)$ transformations on the noncommutative DBI Lagrangian can be captured in the tension of the brane. On the other hand,

we have the original noncommutative DBI theory with the modified tension. This form of the dual theory leads to another solution for the auxiliary gauge field.

In Section 5, the noncommutativity parameter is related to its T -dual and its $SL(2; R)$ dual versions. We shall see that on the effective variables, T -duality and $SL(2; R)$ duality do not commute. In addition, the invariance of the quantity $\frac{G_s}{g_s}$, under the T -duality and $SL(2; R)$ duality will be shown.

In Section 6, we study the Chern–Simons (CS) action. For its commutative theory, for example, see Ref. (Douglas; Li, 1996; Green *et al.*, 1997) and for its noncommutative version, e.g. see Ref. (Mukhi and Suryanarayana, 2000; Mukhi and Suryanarayana; Liu and Michelson, 2001). The effects of the $SL(2; R)$ group on the noncommutative CS action will be studied. We observe that under twice dualization this action remains invariant.

2. NONCOMMUTATIVE DBI LAGRANGIAN AND ITS $SL(2; R)$ DUALITY

Now we study the action of the $SL(2; R)$ group on the noncommutative DBI Lagrangian. We consider an arbitrary Dp -brane. Consider the noncommutative DBI Lagrangian (Seiberg and Witten, 1999)

$$\widehat{\mathcal{L}}_{(0)} = \frac{1}{(2\pi)^p (\alpha')^{\frac{p+1}{2}} G_s^{(0)}} \sqrt{\det(G_{(0)} + 2\pi\alpha'\widehat{F})}, \quad (3)$$

where the index zero shows the cases with zero extra modulus, i.e. $\Phi = 0$. From the definitions of the open string variables $G_{(0)\mu\nu}$, $G_s^{(0)}$ and the noncommutativity parameter $\theta_0^{\mu\nu}$, in terms of the closed string variables $g_{\mu\nu}$, $B_{\mu\nu}$ and g_s (with $\mu, \nu = 0, 1, \dots, p$),

$$\begin{aligned} G_{(0)\mu\nu} &= g_{\mu\nu} - (2\pi\alpha')^2 (B g^{-1} B)_{\mu\nu}, \\ \theta_0^{\mu\nu} &= -(2\pi\alpha')^2 \left(\frac{1}{g + 2\pi\alpha' B} B \frac{1}{g - 2\pi\alpha' B} \right)^{\mu\nu}, \\ G_s^{(0)} &= g_s \left(\frac{\det(g + 2\pi\alpha' B)}{\det g} \right)^{1/2}, \end{aligned} \quad (4)$$

one can find their $SL(2; R)$ transformations. We also require transformation of $\widehat{F}_{\mu\nu}$.

According to the following relation (Seiberg and Witten, 1999)

$$\widehat{F} = (1 + F\theta)^{-1} F, \quad (5)$$

transformation of the noncommutative field strength $\widehat{F}_{\mu\nu}$ can be obtained from the transformations of $\theta^{\mu\nu}$ and the ordinary field strength $F_{\mu\nu}$.

It has been discussed by Townsend (Townsend, 1997) that for D -string there are two $U(1)$ gauge fields $F_{\mu\nu}$ and $\bar{F}_{\mu\nu}$, which form an $SL(2; R)$ doublet, related

to the doublet $\begin{pmatrix} B_{\mu\nu} \\ C_{\mu\nu} \end{pmatrix}$. Also see the Ref. (Townsend, 1992; Bergshoeff *et al.*, 1992). We assume that the field strength $\bar{F}_{\mu\nu}$ can be applied to any Dp -brane. Therefore, $F_{\mu\nu}$ and $\bar{F}_{\mu\nu}$ can be interpreted as DBI fields, but not both simultaneously. Thus, the ordinary gauge field strengths $F_{\mu\nu}$ and $\bar{F}_{\mu\nu}$ transform in the same way as the fields $B_{\mu\nu}$ and $C_{\mu\nu}$,

$$F_{\mu\nu} \rightarrow \tilde{F}_{\mu\nu} = dF_{\mu\nu} - c\bar{F}_{\mu\nu}, \quad \bar{F}_{\mu\nu} \rightarrow \tilde{\bar{F}}_{\mu\nu} = -bF_{\mu\nu} + a\bar{F}_{\mu\nu}. \quad (6)$$

Imposing the $SL(2; R)$ invariance on the ordinary (noncommutative) DBI theory, gives $\tilde{\bar{F}}_{\mu\nu}$ in terms of $F_{\mu\nu}$ ($F_{\mu\nu}$ and $\theta^{\mu\nu}$).

2.1. Commutative Result

Consider the case $C_{\mu\nu} = \frac{d}{c}B_{\mu\nu}$. This means that the field $\tilde{B}_{\mu\nu}$ is zero. In other words, the transformed theory is not noncommutative. Therefore, the $SL(2; R)$ transformation of the Lagrangian (3) reduces to

$$\tilde{\mathcal{L}} = \frac{1}{(2\pi)^p (\alpha')^{\frac{p+1}{2}} \eta^2 g_s} \sqrt{\det(\eta g + 2\pi\alpha'(dF - c\bar{F}))}. \quad (7)$$

For $\bar{F} = \frac{d-\eta}{c}F - \frac{\eta}{c}B$, the Lagrangian (7) is proportional to the DBI Lagrangian, i.e.,

$$\tilde{\mathcal{L}} = \eta^{\frac{p-3}{2}} \mathcal{L}_{DBI}. \quad (8)$$

This equation can be interpreted as follows. The Lagrangian $\tilde{\mathcal{L}}$ describes the same Dp -brane which is described by \mathcal{L}_{DBI} , but with the modified tension

$$\bar{T}_p = \frac{\eta^{(p-3)/2}}{(2\pi)^p (\alpha')^{(p+1)/2} g_s}. \quad (9)$$

For the $D3$ -brane, theory is symmetric, i.e. $\tilde{\mathcal{L}} = \mathcal{L}_{DBI}$, as expected. For the strong coupling of strings $g_s \rightarrow \infty$, the modified tension \bar{T}_p goes to zero. For the weak coupling constant $g_s \rightarrow 0$, this tension for D -particle goes to zero, for D -string is finite and for Dp -brane with $p \geq 2$ approaches to infinity. We also can write

$$\bar{T}_p = \frac{1}{(2\pi)^p (\tilde{\alpha}')^{\frac{p+1}{2}} \tilde{g}_s}, \quad \tilde{\alpha}' = \frac{\alpha'}{\eta}. \quad (10)$$

2.2. Noncommutative Result with $\Phi = 0$

Now we find the conditions for the invariance of the noncommutative theory with zero modulus Φ . Consider the following relations between the scalars and

2-forms

$$e^{2\phi} = \frac{c^2}{1 - (c\chi + d)^2}, \quad C_{\mu\nu} = \frac{d-1}{c} B_{\mu\nu}, \quad (11)$$

which are equivalent to $\eta = 1$ and $\tilde{B}_{\mu\nu} = B_{\mu\nu}$, respectively. These assumptions lead to the relations $\tilde{G}_s^{(0)} = G_s^{(0)}$, $\tilde{G}_{(0)\mu\nu} = G_{(0)\mu\nu}$ and $\tilde{\theta}_0^{\mu\nu} = \theta_0^{\mu\nu}$. In addition, the field strength should be selfdual or anti-selfdual, i.e., $\tilde{F} = \pm \hat{F}$. Therefore, the noncommutative theory (3) becomes $SL(2; R)$ invariant. Since for any matrix M there is $\det M = \det M^T$, the anti-selfdual case for the Lagrangian (3) also is available. In other words, we have $\det(G_{(0)} - 2\pi\alpha'\hat{F}) = \det(G_{(0)} - 2\pi\alpha'\hat{F})^T = \det(G_{(0)} + 2\pi\alpha'\hat{F})$.

According to the Equation (5), the condition on the field strength \hat{F} gives \tilde{F} and consequently \bar{F} in terms of F and θ_0 ,

$$\bar{F} = \frac{d}{c} F \mp \frac{1}{c} [F^{-1} + (1 \mp 1)\theta_0]^{-1}. \quad (12)$$

This is a way to determine the auxiliary field strength $\bar{F}_{\mu\nu}$. For the selfdual case (i.e. the upper signs) the field strength \bar{F} is proportional to F . For the anti-selfdual case (i.e. the lower signs) we have

$$\frac{1}{F} + \frac{1}{\bar{F}} = \frac{2}{-\theta_0^{-1}}. \quad (13)$$

This means that, $-\theta_0^{-1}$ is harmonic mean between F and \bar{F} . The Equation (13) for the commutative case gives an anti-selfdual F , i.e. $\tilde{F} = -F$.

2.3. Noncommutative Result Including Φ

The noncommutative DBI Lagrangian with arbitrary noncommutativity parameter has the dual form

$$\tilde{\mathcal{L}} = \frac{1}{(2\pi)^p (\alpha')^{\frac{p+1}{2}} \tilde{G}_s} \sqrt{\det(\tilde{G} + 2\pi\alpha'(\tilde{\Phi} + \tilde{F}))}, \quad (14)$$

where the effective parameters \tilde{G} , $\tilde{\Phi}$ and \tilde{G}_s have been given by the equations

$$\frac{1}{\tilde{G} + 2\pi\alpha'\tilde{\Phi}} = -\frac{\tilde{\theta}}{2\pi\alpha'} + \frac{1}{\tilde{g} + 2\pi\alpha'\tilde{B}}, \quad (15)$$

$$\tilde{G}_s = \frac{\tilde{g}_s}{\sqrt{\det\left(1 - \frac{\tilde{\theta}}{2\pi\alpha'}(\tilde{g} + 2\pi\alpha'\tilde{B})\right)}}. \quad (16)$$

For the Equations (11) and selfdual θ , the dual Lagrangian (14) is equal to the noncommutative Lagrangian $\widehat{\mathcal{L}}$ (i.e., Equation (14) without tildes) if \widehat{F} is selfdual or

$$\widetilde{\widehat{F}} = -\widehat{F} - 2\Phi . \quad (17)$$

To show invariance under this condition, again use the identity $\det M = \det M^T$.

According to the Equation (17) and dual form of the Equation (5), the field strength \bar{F} is

$$\bar{F} = \frac{d}{c}F + \frac{1}{c}\omega(1 + \theta\omega)^{-1} , \quad (18)$$

where the matrix ω is

$$\omega = (1 + F\theta)^{-1}(F(1 + 2\theta\Phi) + 2\Phi) . \quad (19)$$

As expected, the Equation (18) for $\Phi = 0$ reduces to the Equation (12) with plus signs.

3. DUALITY OF THE DUAL THEORIES

Define the matrix $\tilde{\Lambda}$ as

$$\tilde{\Lambda} \equiv \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \Lambda^{-1} . \quad (20)$$

Therefore, we can write

$$\lambda = \frac{\tilde{a}\tilde{\lambda} + \tilde{b}}{\tilde{c}\tilde{\lambda} + \tilde{d}} , \quad \begin{pmatrix} B_{\mu\nu} \\ C_{\mu\nu} \end{pmatrix} = (\tilde{\Lambda}^T)^{-1} \begin{pmatrix} \tilde{B}_{\mu\nu} \\ \tilde{C}_{\mu\nu} \end{pmatrix} . \quad (21)$$

Also let the parameter $\tilde{\eta}$ be

$$\tilde{\eta} \equiv |\tilde{c}\tilde{\lambda} + \tilde{d}| = \frac{1}{\eta} . \quad (22)$$

This gives

$$g_s = \tilde{\eta}^2 \tilde{g}_s , \quad g_{\mu\nu} = \tilde{\eta} \tilde{g}_{\mu\nu} . \quad (23)$$

That is, in some equations if we change the dual quantities with the initial quantities the resulted equations also hold. With this rule, the Equations (21) and (23) directly can be obtained from the Equations (1) and (2).

For generalization of the above rule let the 2-form $C_{\mu\nu}$ be proportional to $B_{\mu\nu}$ as in the following

$$C_{\mu\nu} = \frac{d - \eta}{c} B_{\mu\nu}. \quad (24)$$

This leads to the relation

$$\tilde{B}_{\mu\nu} = \eta B_{\mu\nu}, \quad (25)$$

or equivalently $\tilde{C}_{\mu\nu} = \frac{1-a\eta}{d-\eta} C_{\mu\nu}$. These equations also hold under the exchange of the dual quantities with the initial quantities. In other words, we have $(\tilde{d} - \tilde{\eta})\tilde{B}_{\mu\nu} = \tilde{c}\tilde{C}_{\mu\nu}$, $B_{\mu\nu} = \tilde{\eta}\tilde{B}_{\mu\nu}$ and $C_{\mu\nu} = \frac{1-\tilde{a}\tilde{\eta}}{\tilde{d}-\tilde{\eta}}\tilde{C}_{\mu\nu}$.

According to the Equations (2), (4) and (25), for the zero modulus Φ , the transformations of $G_{(0)}$, θ_0 and $G_s^{(0)}$ are as in the following

$$\tilde{G}_{(0)\mu\nu} = \eta G_{(0)\mu\nu}, \quad \tilde{\theta}_0^{\mu\nu} = \frac{\theta_0^{\mu\nu}}{\eta}, \quad \tilde{G}_s^{(0)} = \eta^2 G_s^{(0)}. \quad (26)$$

On the other hand, these equations also obey from the above rule.

Since $\tilde{\Lambda} \in SL(2; R)$, we conclude that the initial theory also is $SL(2; R)$ transformed of the dual theory. Therefore, the mentioned rule can be written as

$$\begin{aligned} \text{Initial theory} &\xrightarrow{\Lambda} SL(2; R) \text{ dual theory,} \\ SL(2; R) \text{ dual theory} &\xrightarrow{\tilde{\Lambda}} \text{Initial theory.} \end{aligned} \quad (27)$$

In other words, twice dualization leaves the theory (and related equations) invariant. Note that “*initial theory*” refers to the type IIB theory or DBI theory. In the next sections, we shall see that the rule (27) will be repeated. For example, it also holds for the noncommutative DBI theory, ordinary and noncommutative Chern–Simons actions. The statement (27) for the ordinary DBI theory is obvious, i.e.,

$$\tilde{\mathcal{L}}_{DBI} = \mathcal{L}_{DBI}. \quad (28)$$

4. RELATIONS BETWEEN THE EFFECTIVE VARIABLES

The noncommutative DBI Lagrangian and the $SL(2; R)$ duality of it can be described more generally, such that the noncommutativity parameters θ and $\tilde{\theta}$ become arbitrary (Seiberg and Witten, 1999). Therefore, the extra moduli Φ and $\tilde{\Phi}$ are not zero (for example, the dual theory was given by the Equation (14)). The Equations (26) guide us to introduce the following relations between the effective metrics and the extra moduli

$$\tilde{G}_{\mu\nu} = \eta G_{\mu\nu}, \quad \tilde{\Phi}_{\mu\nu} = \eta \Phi_{\mu\nu}. \quad (29)$$

According to the Equations (15) and (29) we obtain

$$\tilde{\theta}^{\mu\nu} = \frac{\theta^{\mu\nu}}{\eta}. \quad (30)$$

This implies that, if the effective theory is noncommutative (ordinary) the dual theory of it also is noncommutative (ordinary). Note that if we introduce the Equation (30) then we obtain the Equations (29). The Equations (16) and (30) give the following relation between the effective string couplings \tilde{G}_s and G_s ,

$$\tilde{G}_s = \eta^2 G_s. \quad (31)$$

The Equations (29)–(31) have the following properties. (a) They are consistent with the rule (27). In other words, they can be written in the forms $G_{\mu\nu} = \tilde{\eta}\tilde{G}_{\mu\nu}$, $\Phi_{\mu\nu} = \tilde{\eta}\tilde{\Phi}_{\mu\nu}$, $\theta^{\mu\nu} = \frac{\tilde{\theta}^{\mu\nu}}{\tilde{\eta}}$ and $G_s = \tilde{\eta}^2\tilde{G}_s$. (b) For the commutative case, i.e., $\theta = 0$ we have $\tilde{\theta} = 0$. Thus, the Equations (29) change to $\tilde{g}_{\mu\nu} = \eta g_{\mu\nu}$ and $\tilde{B}_{\mu\nu} = \eta B_{\mu\nu}$. (c) For the variables

$$\begin{aligned} \theta &= B^{-1}, & G &= -(2\pi\alpha')^2 B g^{-1} B, & \Phi &= -B, \\ \tilde{\theta} &= \tilde{B}^{-1}, & \tilde{G} &= -(2\pi\alpha')^2 \tilde{B} \tilde{g}^{-1} \tilde{B}, & \tilde{\Phi} &= -\tilde{B} \end{aligned} \quad (32)$$

the Equations (29) and (30) reduce to identities. Note that these variables also satisfy the Equation (15) and this equation without tildes. (d) For $\theta = \theta_0$ the Equation (30) gives $\tilde{\theta} = \tilde{\theta}_0$, therefore, $\Phi = \tilde{\Phi} = 0$. In this case as expected, there are $G = G_{(0)}$, $\tilde{G} = \tilde{G}_{(0)}$, $G_s = G_s^{(0)}$ and $\tilde{G}_s = \tilde{G}_s^{(0)}$. On the other hand, the Equations (29)–(31) reduce to the results (26).

The Equations (25), (29), (30) and the second Equation of (2) lead to the relations

$$\tilde{Q}_{(1)}^{\mu\nu} \tilde{Q}_{(2)\rho\sigma} = Q_{(1)}^{\mu\nu} Q_{(2)\rho\sigma}, \quad \tilde{Q}_{(1)}^{\mu\nu} Q_{(2)\rho\sigma} = Q_{(1)}^{\mu\nu} \tilde{Q}_{(2)\rho\sigma}, \quad (33)$$

where $Q_1, Q_2 \in \{G, \Phi, \theta, g, B\}$. That is, the quantities $Q_{(1)}^{\mu\nu} Q_{(2)\rho\sigma}$ and $\tilde{Q}_{(1)}^{\mu\nu} Q_{(2)\rho\sigma}$ are $SL(2; R)$ invariant. On the other hand, since we have $\tilde{Q}_{(i)} = Q_{(i)}$ for $i = 1, 2$, the Equations (33) are consistent with the rule (27).

According to the Equations (29) and (31), for the following values of the dual field $\tilde{\tilde{F}}$,

$$\tilde{\tilde{F}} = \eta[\pm\hat{F} - (1 \mp 1)\Phi], \quad (34)$$

the dual Lagrangian (14) takes the form

$$\tilde{\tilde{\mathcal{L}}} = \eta \frac{p-3}{2} \hat{\mathcal{L}}. \quad (35)$$

After substituting (34) in (14), for the lower signs one should perform transpose on the matrices in (14). Again use the identity $\det M = \det M^T$. This Lagrangian describes the same noncommutative Dp -brane, which also is given by $\hat{\mathcal{L}}$, but with

the modified tension, i.e., $\widehat{T}_p = \eta^{\frac{p-3}{2}} \widehat{T}_p$. For the $D3$ -brane the theory is invariant. For the commutative case, the Equation (35) reduces to the Equation (8), as expected.

The Equation (34) with lower signs, is generalization of the Equation (17). The field strength \bar{F} , extracted from the Equation (34), is

$$\bar{F} = \frac{d}{c}F + \frac{1}{c}\Omega \left(1 + \frac{1}{\eta}\theta\Omega\right)^{-1}, \quad (36)$$

where the matrix Ω is

$$\Omega = \eta(1 + F\theta)^{-1}(F[\mp 1 + (1 \mp 1)\theta\Phi] + (1 \mp 1)\Phi). \quad (37)$$

Since the Equation (34) can be written as $\widehat{F} = \bar{\eta}[\pm\widehat{F} - (1 \mp 1)\widehat{\Phi}]$, this with the Equations (29)–(31) imply that, in the rule (27) the “initial theory” also can be the noncommutative DBI theory. This also can be seen from the Equation (35), i.e. $\widehat{\mathcal{L}} = \bar{\eta}^{\frac{p-3}{2}}\widetilde{\mathcal{L}}$, or equivalently

$$\widetilde{\mathcal{L}} = \widehat{\mathcal{L}}. \quad (38)$$

5. COMMUTATOR OF THE $SL(2; R)$ DUALITY AND T -DUALITY ON THE EFFECTIVE VARIABLES

Previously we have observed that the effective metric $G_{(0)\mu\nu}$ and the non-commutativity parameter $\theta_0^{\mu\nu}$ under the T -duality transform to $G'_{(0)\mu\nu}$ and $\theta_0^{\prime\mu\nu}$ as in the following (Kamani, 2002)

$$G'_{(0)\mu\nu} = g_{\mu\nu}, \quad \theta_0^{\prime\mu\nu} = (2\pi\alpha')^2(B^{-1})^{\mu\nu}. \quad (39)$$

The actions of $SL(2; R)$ duality on these equations give

$$(\widetilde{G}'_0)_{\mu\nu} = \eta g_{\mu\nu}, \quad (\widetilde{\theta}'_0)^{\mu\nu} = \frac{1}{\eta}(2\pi\alpha')^2(B^{-1})^{\mu\nu}. \quad (40)$$

Application of T -duality on the first and second equations of (26) and then comparison of the results with the Equations (40), lead to the relations

$$[(\widetilde{G}'_0) - (\widetilde{G}_0)']_{\mu\nu} = (\eta - \eta')G'_{(0)\mu\nu}, \quad [(\widetilde{\theta}'_0) - (\widetilde{\theta}_0)']^{\mu\nu} = \left(\frac{1}{\eta} - \frac{1}{\eta'}\right)\theta_0^{\prime\mu\nu}, \quad (41)$$

where η' is T -duality of η . These equations imply that, on the open string metric and noncommutativity parameter, unless $\eta = \eta'$, T -duality and $SL(2; R)$ duality do not commute with each other. We shall show that for the nonzero modulus Φ , these equations also hold.

In the presence of the extra modulus Φ , we have the following relation (Kamani, 2002)

$$G' + 2\pi\alpha'\Phi' = (g + 2\pi\alpha'B)^{-1}(G - 2\pi\alpha'\Phi)(g - 2\pi\alpha'B)^{-1}. \quad (42)$$

Therefore, there is the following relation between the noncommutativity parameter $\theta^{\mu\nu}$ and its T -duality $\theta'^{\mu\nu}$,

$$\theta'^{\mu\nu} = -[(g - 2\pi\alpha'B)\theta(g + 2\pi\alpha'B)]^{\mu\nu}. \quad (43)$$

According to this equation and Equation (30) we obtain

$$\theta'^{\mu\nu} = -\eta[(g - 2\pi\alpha'B)\tilde{\theta}(g + 2\pi\alpha'B)]^{\mu\nu}. \quad (44)$$

That is, the T -duality and $SL(2; R)$ duality versions of the noncommutativity parameter are related to each other.

Action of the $SL(2; R)$ duality on G' , Φ' and θ' of the Equations (42) and (43) and also action of T -duality on \tilde{G} , $\tilde{\Phi}$ and $\tilde{\theta}$ of the Equations (29) and (30), and then comparison of the results, give

$$[(\tilde{Q}') - (\tilde{Q})]_{\mu\nu} = (\eta - \eta')Q'_{\mu\nu}, \quad [(\tilde{\theta}') - (\tilde{\theta})]^{\mu\nu} = \left(\frac{1}{\eta} - \frac{1}{\eta'}\right)\theta'^{\mu\nu}, \quad (45)$$

where $Q \in \{G, \Phi\}$. Let us denote the dualities of Q as $Q' \equiv TQ$ and $\tilde{Q} \equiv SQ$. Thus, the Equations (45) take the forms

$$([S, T]Q)_{\mu\nu} = (\eta - \eta')(TQ)_{\mu\nu}, \quad ([S, T]\theta)^{\mu\nu} = \left(\frac{1}{\eta} - \frac{1}{\eta'}\right)(T\theta)^{\mu\nu}. \quad (46)$$

Similarly, for the effective string coupling G_s there is

$$[S, T]G_s = (\eta^{(3-p)/2} - \eta'^2)(TG_s). \quad (47)$$

Therefore, on the variables G , Φ , θ and G_s T -duality and $SL(2; R)$ duality do not commute. In other words, the commutator of these dualities, is proportional to the effects of T -duality.

The T -duality of the effective string coupling is $G'_s = \frac{G_s}{\sqrt{\det(g+2\pi\alpha'B)}}$. This implies $\frac{G_s}{g_s}$ is a T -duality invariant quantity (Kamani, 2002). From this and the Equation (31) we conclude that

$$\frac{G'_s}{g'_s} = \frac{\tilde{G}_s}{\tilde{g}_s} = \frac{G_s}{g_s}. \quad (48)$$

That is, the ratio $\frac{G_s}{g_s}$ also is invariant under the $SL(2; R)$ duality. Therefore, on the quantity $\frac{G_s}{g_s}$, T -duality and $SL(2; R)$ duality commute.

6. $SL(2; R)$ DUALITY OF THE NONCOMMUTATIVE CHERN–SIMONS ACTION

The DBI action describes the couplings of a Dp -brane to the massless Neveu–Schwarz fields $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ . The interactions with the massless Ramond–Ramond (R-R) fields are incorporated in the Chern–Simons action (Douglas; Li, 1996; Green *et al.*, 1997)

$$S_{CS} = \frac{1}{(2\pi)^p (\alpha')^{(p+1)/2} g_s} \int \sum_n C^{(n)} \wedge e^{2\pi\alpha'(B+F)}, \quad (49)$$

where $C^{(n)}$ denotes the n -form R-R potential. The exponential should be expanded so that the total forms have the rank of the worldvolume of brane. In fact, this action is for a single BPS Dp -brane.

The noncommutative Chern–Simons action for constant fields can be written as in the following (Mukhi and Suryanarayana, 2000; Mukhi and Suryanarayana)

$$\begin{aligned} \widehat{S}_{CS} &= \frac{1}{(2\pi)^p (\alpha')^{(p+1)/2} g_s} \int \sqrt{\det(1 - \theta \widehat{F})} \\ &\times \sum_n C^{(n)} \wedge \exp(2\pi\alpha'[B + \widehat{F}(1 - \theta \widehat{F})^{-1}]), \end{aligned} \quad (50)$$

also see Ref. (Liu and Michelson, 2001). This action holds for general modulus Φ . It describes the R-R couplings to a noncommutative Dp -brane.

Now we study the effects of the $SL(2; R)$ group on this action. We can apply \widetilde{F} from (34). For simplicity, choose the upper signs for \widetilde{F} . In addition, the Equations (25) and (30) can be used for \widetilde{B} and $\widetilde{\theta}$. Adding all these together, we obtain

$$\begin{aligned} \widetilde{S}_{CS} &= \frac{1}{(2\pi)^p (\alpha')^{(p+1)/2} \eta^2 g_s} \int \sqrt{\det(1 - \theta \widehat{F})} \\ &\times \sum_n \widetilde{C}^{(n)} \wedge \exp(2\pi\alpha'\eta[B + \widehat{F}(1 - \theta \widehat{F})^{-1}]). \end{aligned} \quad (51)$$

Therefore, we should determine the dual fields $\{\widetilde{C}^{(n)}\}$. Since our attention is on the type IIB theory, $\widetilde{C}^{(n)}$ is an even form. The dual fields $\widetilde{C}^{(0)} \equiv \widetilde{\chi}$ and $\widetilde{C}^{(2)} \equiv \widetilde{C}$ have been given by the transformations (1). The field $C^{(4)}$ corresponds to the $D3$ -brane. It was shown in (Tseytlin, 1996; Green and Gutperle, 1996) that the invariance of the equations of motion, extracted from the total action $S_{DBI} + S_{CS}$, under the $SL(2; R)$ group, gives the transformations (1) and

$$C^{(4)} \rightarrow \widetilde{C}^{(4)} = C^{(4)}. \quad (52)$$

For the forms $\widetilde{C}^{(6)}$, $\widetilde{C}^{(8)}$ and $\widetilde{C}^{(10)}$ one may use the Hodge duals of the forms $\widetilde{C}^{(4)}$, $\widetilde{C}^{(2)}$ and $\widetilde{C}^{(0)}$, which are available. However, we have the following results at least for $n \leq 4$.

The noncommutative Chern–Simons action (50) respects the rule (27), if twice dualization of the R-R fields are invariant

$$\widetilde{\widetilde{C}}^{(n)} = C^{(n)}. \quad (53)$$

From the transformations (1) and (52) explicitly one can see this equation for $C^{(0)}$, $C^{(2)}$ and $C^{(4)}$. On the other hand, using (53) (at least for $n \leq 4$) and then applying $SL(2; R)$ transformations on the dual action (51), we obtain

$$\widetilde{\widetilde{S}}_{CS} = \widehat{S}_{CS}. \quad (54)$$

From the Equations (1), (2), (6) and (20) we have $\widetilde{\widetilde{B}}_{\mu\nu} = B_{\mu\nu}$, $\widetilde{\widetilde{g}}_s = g_s$ and $\widetilde{\widetilde{F}}_{\mu\nu} = F_{\mu\nu}$. By considering the Equation (53), we observe that the ordinary Chern–Simons action (49) also obey the rule (27),

$$\widetilde{\widetilde{S}}_{CS} = S_{CS}. \quad (55)$$

For vanishing noncommutativity parameter, the Equation (54) reduces to (55), as expected.

7. CONCLUSIONS

We studied the action of the $SL(2; R)$ group on the noncommutative DBI theory with zero and nonzero extra modulus Φ . The invariance of the theory determines the corresponding noncommutative field strength $\widehat{F}_{\mu\nu}$. As a consequence, the auxiliary field strength $\widetilde{F}_{\mu\nu}$ has been obtained. For a special value of the R-R 2-form, the $SL(2; R)$ group on the noncommutative DBI Lagrangian produces a theory which describes an ordinary brane with the modified tension. For the $D3$ -brane the resulted ordinary theory is DBI theory, as expected.

We observed that the extracted equations of the ordinary DBI and noncommutative DBI theories under the exchange of the variables with their dual variables are invariant. In other words, twice dualizing of these theories and the corresponding variables and equations, does not change them. This implies that these theories and their $SL(2; R)$ transformations, are dual of each other.

By introducing some relations (which are consistent with the rule (27)) between the effective variables and their duals, we obtained some other equations that are $SL(2; R)$ invariant. Therefore, another solution for the auxiliary gauge field was found. In this case, $SL(2; R)$ duality of the noncommutative DBI theory is proportional to the noncommutative DBI theory. For the $D3$ -brane the theory is selfdual.

We showed that the noncommutativity parameter, its T -dual and its $SL(2; R)$ dual have relations with each other. We found that on the open string metric, noncommutativity parameter, the extra modulus Φ and the effective string coupling, T -duality and $SL(2; R)$ duality do not commute. We also observed that the ratio

of the effective string coupling to the string coupling under the above dualities is invariant.

Finally, we studied the effects of the $SL(2; R)$ group on the noncommutative Chern–Simons action. Under two successive dualizations, similar the DBI theory, this action remains invariant. This also occurs for the ordinary Chern–Simons action.

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